Thus the computed chi-square value is 5.53, with $\delta = k - 1 - m = 7 - 1 - 2 = 4$. This fit is good at both the 5 percent and 1 percent levels of significance, since $P(\chi_4^2 > 9.49) = 0.05$ and $P(\chi_4^2 > 11.14) = 0.01$.

In concluding this section, it is to be noted that chi-square tests on goodness-of-fit are typically single-tail tests. Either the computed chi-square value is too large to be attributable to chance, the fit is considered too bad, and the hypothesis is rejected; or, the computed chi-square value is not too large, the fit is considered good, and the hypothesis is accepted. However, this procedure should not obscure the fact that in effect there are two admissible alternative hypotheses: It is possible that the fit may be too good; that is, the differences between observed and expected frequencies may be smaller than those that could result from random sampling. Therefore, it has been suggested that chi-square tests on goodness-of-fit should be two-sided. Although we do not recommend this procedure, the student should always bear in mind that, in repeated sampling, too good a fit is just as likely as too bad a fit. When the computed chi-square value is too close to zero, we should suspect the possibility that the two sets of frequencies have been manipulated in order to force them to agree and, therefore, the design of our experiment should be thoroughly checked.

14.8 Tests of Independence: Contingency Table Tests

Tests on frequencies and on goodness-of-fit are both concerned with multinomial populations. In both cases, populations as well as samples are classified in accordance with a single attribute. We shall now see that the same technique of chi-square tests can be applied to data other than multinomial distributions.

When the population and the sample are classified according to two or more attributes, we may use *tests of independence* to determine whether the principles or criteria employed for cross classification are meaningful or effective. For instance, a random sample of n retail stores may be classified by size of capitalization and by type of ownership. The proportion of each of the two classes in the population is unknown. Our interest would be to establish whether there is any dependency relationship between a store's capitalization and its type of ownership. Clearly, in a case such as this, we would want to test the hypothesis that the size of capitalization is independent of the type of ownership against the hypothesis that they are related. In this test of independence, we try to verify the significance of a capitalization difference in type of ownership. If the difference is significant, we conclude that the type of ownership is independent of the size of capitalization. Otherwise, we say that these two criteria of classification are related or dependent.

Tests of independence are also called *contingency table tests*. So far, we have been concerned with only *one-way classification tables* since, in each case, the

From "Statistical Analysis" by Ya-Lun Chou, Holt, Reinhart and Winston, 1969.

observed frequencies have occupied a single row or a single column. Also because the observed frequencies are distributed in k classes (columns or rows), one-way classification tables are called *1 × k* (read *1* by k) or *k × 1 tables*. Extending these ideas, we can arrive at 2 × 2, or *two-way classification tables* and, generally, $R \times C$ tables in which the observed frequencies occupy R rows and C columns. Such tables are often called *contingency tables*. Corresponding to each observed frequency in an $R \times C$ table, there is an expected frequency computed under the specified null hypothesis. Frequencies, observed or expected, which occupy the cells of a contingency table are called *cell frequencies*. The total frequency in each row or each column is called the *marginal frequency*.

To evaluate differences between observed and expected frequencies contained in contingency tables, we employ the same statistic as that for tests discussed in the previous section,

$$\chi_s^2 = \sum \frac{(o_i - e_i)^2}{e_i}$$

The sum is taken over all cells in a contingency table and, in general,

$$\delta = (R - 1)(C - 1)$$

In testing independence with a 2 × 2 contingency table, we would have $\delta = (2 - 1)(2 - 1) = 1$. As recommended before, the continuity correction factor of ½ should be used to compute the chi-square value. This is especially important with small samples so that some or all expected frequencies are less than 5. For large samples, corrected and uncorrected chi-square values may be practically the same and, as a consequence, the continuity correction factor may be ignored. Let us consider an example.

There is reason to believe that high-income families usually send their children to private colleges and low-income families often send their children to city or state colleges. To verify this, 1600 families are selected at random from California and the following results are obtained:

OBSERVED DATA

	College		
Income	*Private*	*Public*	Total
Low	506	494	1000
High	438	162	600
Total	944	656	1600

Here, we have two classifications: income, and type of college. At a glance, we see that, relatively, a greater number of high-income families send their children to private colleges than do low-income families. But is such a difference in proportions significant or is it merely the result of chance variations in random sampling? The chi-square test of independence can be used to answer questions

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Under the assumption that family income and type of college are independent, we would then expect the proportion of all families which send their children to private colleges to be $944/1600$. The expected number of low-income families which send their children to private colleges is then

$$(944/1600)(1000) = 590$$

Now, since the expected frequencies must agree with the marginal frequencies of the observed frequencies, and since there are four cells in a 2×2 contingency table, we can automatically fill in the remaining three expected numbers once any one of the four expected numbers is determined. It is for this reason that there is only 1 degree of freedom associated with a 2×2 contingency table. The expected frequencies for our example are as follows:

EXPECTED DATA

	College		
Income	*Private*	*Public*	Total
Low	590	410	1000
High	354	246	600
Total	944	656	1600

Following our general testing procedure, the chi-square test of independence for data on hand becomes

1. **Hypotheses:** H_0: Income and type of college are independent
H_1: They are related or dependent

2. **Level of Significance:** $\alpha = 0.01$

3. **Testing Statistic:** $\sum \frac{(o_i - e_i)^2}{e_i}$, which is approximately distributed as χ_1^2

4. **Decision Rule:** The critical region of this test is $6.63 < \chi_1^2 < \infty$. Thus, accept H_0 if and only if the computed chi-square value is less than 6.63.

5. **Computations:** From observed and expected data given before, we have

$$\chi_1^2 = \sum \frac{(o_i - e_i)^2}{e_i} = \frac{(506 - 590)^2}{590} + \frac{(494 - 410)^2}{410} + \frac{(438 - 354)^2}{354} + \frac{(162 - 246)^2}{246} = 77.78$$

6. **Decisions:** Reject H_0. Family income and type of college are not independent. A greater proportion of high-income families send their children to private colleges.

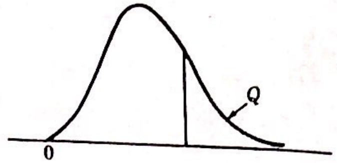

Table III. Upper Percentage Points of the χ^2 Distribution

ν \ Q	0.995	0.990	0.975	0.950	0.900	0.750	0.500
1	$392704 \cdot 10^{-10}$	$157088 \cdot 10^{-9}$	$982069 \cdot 10^{-9}$	$393214 \cdot 10^{-8}$	0.0157908	0.1015308	0.454937
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	0.575364	1.38629
3	0.0717212	0.114832	0.215795	0.351846	0.584375	1.212534	2.36597
4	0.206990	0.297110	0.484419	0.710721	1.063623	1.92255	3.35670
5	0.411740	0.554300	0.831211	1.145476	1.61031	2.67460	4.35146
6	0.675727	0.872085	1.237347	1.63539	2.20413	3.45460	5.34812
7	0.989265	1.239043	1.68987	2.16735	2.83311	4.25485	6.34581
8	1.344419	1.646482	2.17973	2.73264	3.48954	5.07064	7.34412
9	1.734926	2.087912	2.70039	3.32511	4.16816	5.89883	8.34283
10	2.15585	2.55821	3.24697	3.94030	4.86518	6.73720	9.34182
11	2.60321	3.05347	3.81575	4.57481	5.57779	7.58412	10.3410
12	3.07382	3.57056	4.40379	5.22603	6.30380	8.43842	11.3403
13	3.56503	4.10691	5.00874	5.89186	7.04150	9.29906	12.3398
14	4.07468	4.66043	5.62872	6.57063	7.78953	10.1653	13.3393
15	4.60094	5.22935	6.26214	7.26094	8.54675	11.0365	14.3389
16	5.14224	5.81221	6.90766	7.96164	9.31223	11.9122	15.3385
17	5.69724	6.40776	7.56418	8.67176	10.0852	12.7919	16.3381
18	6.26481	7.01491	8.23075	9.39046	10.8649	13.6753	17.3379
19	6.84398	7.63273	8.90655	10.1170	11.6509	14.5620	18.3376
20	7.43386	8.26040	9.59083	10.8508	12.4426	15.4518	19.3374
21	8.03366	8.89720	10.28293	11.5913	13.2396	16.3444	20.3372
22	8.64272	9.54249	10.9823	12.3380	14.0415	17.2396	21.3370
23	9.26042	10.19567	11.6885	13.0905	14.8479	18.1373	22.3369
24	9.88623	10.8564	12.4011	13.8484	15.6587	19.0372	23.3367
25	10.5197	11.5240	13.1197	14.6114	16.4734	19.9393	24.3366
26	11.1603	12.1981	13.8439	15.3791	17.2919	20.8434	25.3364
27	11.8076	12.8786	14.5733	16.1513	18.1138	21.7494	26.3363
28	12.4613	13.5648	15.3079	16.9279	18.9392	22.6572	27.3363
29	13.1211	14.2565	16.0471	17.7083	19.7677	23.5666	28.3362
30	13.7867	14.9535	16.7908	18.4926	20.5992	24.4776	29.3360
40	20.7065	22.1643	24.4331	26.5093	29.0505	33.6603	39.3354
50	27.9907	29.7067	32.3574	34.7642	37.6886	42.9421	49.3349
60	35.5346	37.4848	40.4817	43.1879	46.4589	52.2938	59.3347
70	43.2752	45.4418	48.7576	51.7393	55.3290	61.6983	69.3344
80	51.1720	53.5400	57.1532	60.3915	64.2778	71.1445	79.3343
90	59.1963	61.7541	65.6466	69.1260	73.2912	80.6247	89.3342
100	67.3276	70.0648	74.2219	77.9295	82.3581	90.1332	99.3341
z_Q	−2.5758	−2.3263	−1.9600	−1.6449	−1.2816	−0.6745	0.0000

Table III (continued)

ν \ Q	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944	10.828
2	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966	13.816
3	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381	16.266
4	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602	18.467
5	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496	20.515
6	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476	22.458
7	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777	24.322
8	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550	26.125
9	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893	27.877
10	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882	29.588
11	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569	31.264
12	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995	32.909
13	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194	34.528
14	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193	36.123
15	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013	37.697
16	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672	39.252
17	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185	40.790
18	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564	42.312
19	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822	43.820
20	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968	45.315
21	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010	46.797
22	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956	48.268
23	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813	49.728
24	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585	51.179
25	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278	52.620
26	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899	54.052
27	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449	55.476
28	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933	56.892
29	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356	58.302
30	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720	59.703
40	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659	73.402
50	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900	86.661
60	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517	99.607
70	77.5766	85.5271	90.5312	95.0231	100.425	104.215	112.317
80	88.1303	96.5782	101.879	106.629	112.329	116.321	124.839
90	98.6499	107.565	113.145	118.136	124.116	128.299	137.208
100	109.141	118.498	124.342	129.561	135.807	140.169	149.449
z_Q	+0.6745	+1.2816	+1.6449	+1.9600	+2.3263	+2.5758	+3.0902

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